PMM U.S.S.R.,Vol.45,pp.199-207 0021-8928/82/2 0199 $7.50/0

UDC 532.592

WIND-INDUCED PERTURBATIONS OF THE SURFACE OF A VISCOUS FLUID*

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Interaction between a heavy incompressible fluid and air flow is considered in two cases: 1) the air flow velocity is specified at some height above the water surface, and 2) the layer of fluid of depth h_1 is subjected to constant shear stresses S' generated at its surface by the air flow. In the first case integral representations of the form of the fluid surface are obtained in linear formulation. Asymptotic analysis of these is carried out for small $\nu g V^{-3}$ and limited time intervals (V is the scale of the air flow velocity). Some qualitative conclusions are arrived at relative to the initial stages of interaction between the air flow and the fluid surface. In the second case when $S'h_1^2/(\rho\nu^2) \to \infty$, the auto-oscillation mode in the fluid layer the loss of stability of the steady mode with triangular velocity profile is investigated. Sufficient conditions of stability of such auto-oscillation mode are obtained and some of its properties investigated.

Wind-induced waves in a perfect fluid are simulated on the basis of energy and statistical considerations /1/. If viscous friction is taken into account, surface waves can be considered as the result of application to the fluid surface of tangential and normal stresses; such problems were described in /2/ in terms of linearized Navier—Stokes equations, and in /3/ in nonlinear formulation. Below, an attempt is made to explain the generation of waves by the interaction between the water surface and a given atmospheric flow. The stable wave mode at the fluid surface induced by subjecting it to constant shear stresses is studied.

1. We introduce the Cartesian coordinates x', y', z' with the Oz'-axis in the opposite direction to that of the force of gravity. Let a stationary heavy incompressible fluid ("water") of density ρ_2 and viscosity ν_2 occupy the half-space $z' < 0$ up to instant $t' = 0$, and a stationary heavy fluid ("air") with properties $\rho_1 < \rho_2$ and ν_1 occupy the half-space $z' > 0$. We assume that in part $z' \geqslant h'$ of the "atmosphere" a flow of air is maintained at velocity $v'(x', t')$ parallel to the $Ox'y'$ plane along the Ox'-axis for $t' > 0$. The height h' of the /air/ flow above the plane $z' = 0$ depends on the state of the ocean surface. Experiments /1/ show that at approximately 14 cm above the wave crest the wind velocity is virtually unperturbed by the waves generated by it. This implies that height h' increases with the amplitude of wind induced waves. For simplicity we assume h' to be constant. The initial ocean surface form $z' = 0$ varies under the action of air flow and assumes the form $z' = \zeta'(x', t)$. Flows $v_x^{(2)}(x', z', t'), v_z^{(2)}$ and $v_x^{(1)}, v_z^{(1)}$ appear in the ocean $z' < \zeta'$ and in the atmosphere layer $\zeta' < z' < h'$ (the superscript unity denotes characteristics of atmospheric flows). For the determination of wind-induced perturbations $v_x^{(j)}, v_z^{(j)}$ and hydrodynamic pressures $p_1^{(j)}$ $(j = 1, 2)$ generated by them we use the system of linearized Navier—Stokes equations, with the conjugation conditions for flows in the ocean and the atmosphere satisfied not on the unknown surface $z' = \zeta'$ but in the plane $z' = 0$. Then in the dimensionless variables

$$x = x'\sqrt{\frac{g}{\nu_2 V}}, \quad z = z'\sqrt{\frac{g}{\nu_2 V}}, \quad t = t'\sqrt{\frac{Vg}{\nu_2}}, \quad h = h'\sqrt{\frac{g}{\nu_2 V}} \tag{1.1}$$

$$v_x^{(j)} = Vu^{(j)}, \quad v_z^{(j)} = Vw^{(j)}, \quad \zeta' = \zeta\sqrt{\frac{\nu_2 V}{g}}, \quad p_1^{(j)} = q_1^{(j)} + \rho_j g z'$$

$$q_1^{(j)} = \rho_j V^2 q^{(j)}; \quad \varepsilon = \varepsilon_2 = \sqrt{\frac{\nu_2 g}{V_3}}, \quad \varepsilon_1 = \nu\varepsilon, \quad \rho = \frac{\rho_1}{\rho_2}, \quad \nu = \frac{\nu_1}{\nu_2}$$

$$\mu = \rho\nu, \quad v(x, t) = \frac{v'}{V}, \quad V = \max_{x', t'} |v'(x', t')|$$

the system of equations and boundary conditions for the determination of $\mathbf{v}^{(j)} = \{u^{(j)}, w^{(j)}\}$, $q^{(j)}$ and ζ is of the form

$$\frac{\partial \mathbf{v}^{(j)}}{\partial t} + \nabla q^{(j)} = \varepsilon_j \nabla^2 \mathbf{v}^{(j)}, \quad \nabla \mathbf{v}^{(j)} = 0; \quad \zeta = \mathbf{v}^{(j)} = 0\ (t = 0), \quad j = 1, 2$$

$$u^{(1)} = v(x, t), \quad w^{(1)} = 0\ (z = h); \quad u^{(2)} = w^{(2)} = 0 \quad (z = -\infty)$$

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*Prikl.Matem.Mekhan.,45,No.2,275-286,1981

$$\frac{\partial \zeta}{\partial t} = w^{(1)}|_{z=+0} = w^{(2)}|_{z=-0}, \quad \frac{\partial u^{(2)}}{\partial z} + \frac{\partial w^{(2)}}{\partial x}\Big|_{z=-0} = \mu\left(\frac{\partial u^{(1)}}{\partial z} + \frac{\partial w^{(1)}}{\partial x}\right)_{z=+0}$$

$$\varepsilon(1-\rho)\zeta = q^{(2)} - 2\varepsilon\frac{\partial w^{(2)}}{\partial z}\Big|_{z=-0} - \rho q^{(1)} + 2\mu\varepsilon\frac{\partial w^{(1)}}{\partial z}\Big|_{z=+0}, \quad u^{(1)}|_{z=+0} = u^{(2)}|_{z=-0}$$

To solve this problem we apply the Laplace–Carson transform with respect to t and the Fourier transform with respect to x. The Laplace–Carson and the Fourier transforms will be denoted, respectively, by a bar and a capital letter that corresponds to the original. Omitting intermediate operations we present the equation

$$\bar{Z} = \frac{i\rho s\bar{V}\,\mathrm{sgn}\,\omega\exp\left(-h\sqrt{\omega^2+\frac{s}{\nu\varepsilon}}\right)}{[2\varepsilon\omega^2(1-\mu)+s]^2+|\omega|\varepsilon}[1+O(\delta)], \quad s\not\to 0 \tag{1.2}$$

$$\delta = \max(\rho^2, \rho\sqrt{\varepsilon}, \varepsilon^2), \quad \mathrm{Re}\sqrt{s}>0, \quad \bar{V} = \frac{s}{\sqrt{2\pi}}\int_0^\infty\int_{-\infty}^\infty v(x,t)\,e^{-st-i\omega x}\,dx$$

for the image of the interface $\bar{Z}$ of the fluids, which holds for small values of parameters ρ and ε. The constraint $s\not\to 0$ is due to the omission of terms of order $O(\exp(\pm|\omega h|))$ but not of order $O(\exp(\pm h\sqrt{\omega^2+s/(\nu\varepsilon)}))$. This simplification is valid for small ε but not when $s\to 0$. Hence results obtained below become invalid as $t\to\infty$.

Converting (1.2) /4/ and omitting $O(\delta)$, we obtain the expression for the shape of the two fluid media interface which is valid under the condition that the quantity of order $O(\delta)$ is small in comparison with unity

$$\zeta = \frac{\rho}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_0^t\int_{-\infty}^\infty V(\omega, t-\tau)\,e^{i\omega x}K(\omega,\tau)\,d\omega\,d\tau \tag{1.3}$$

$$K(\omega,t) = \frac{1}{4\sqrt{|\omega|\varepsilon}}\sum_{n=1}^{2}(-1)^{n+1}\exp(s_n t)\times$$

$$\left\{\exp\left(-h\sqrt{\omega^2+\frac{s_n}{\nu\varepsilon}}\right)\mathrm{erfc}\left[\frac{h}{2\sqrt{\nu\varepsilon t}}-\sqrt{(\nu\varepsilon\omega^2+s_n)t}\right]+\right.$$

$$\left.\exp\left(h\sqrt{\omega^2+\frac{s_n}{\nu\varepsilon}}\right)\mathrm{erfc}\left[\frac{h}{2\sqrt{\nu\varepsilon t}}+\sqrt{(\nu\varepsilon\omega^2+s_n)t}\right]\right\}\mathrm{sgn}\,\omega$$

$$s_n = -2\varepsilon\omega^2(1-\mu) - i(-1)^n\sqrt{|\omega|\varepsilon}.$$

On the basis of the smallness of ε, formulas (1.3) can be simplified by the use of asymptotic formulas for function $\mathrm{erfc}\,z$ as $|z|\to\infty$ /5/. According to these formulas the basic contribution to the asymptotic expression (1.3) as $\varepsilon\to 0$ is provided by those values of the argument of functions erfc which lie in the left-hand half-plane. These values are determined by solving for ω the inequality

$$\mathrm{Re}\left[\frac{h}{2\sqrt{\nu\varepsilon t}}-\sqrt{(\nu\varepsilon\omega^2+s_n)t}\right]<0$$

It appears that when

$$|\omega| > \omega_0(t) = \frac{h}{2\varepsilon t\sqrt{\nu k}} + O(\varepsilon), \quad k = \nu - 2(1-\mu) \tag{1.4}$$

then

$$\zeta = \frac{i\rho}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_0^t\int_{|\omega|>\omega_0(\tau)} V(\omega, t-\tau)\,K(\omega,\tau,x)\,d\omega\,d\tau \tag{1.5}$$

$$K(\omega,t,x) = \frac{\mathrm{sgn}\,\omega}{\sqrt{\varepsilon|\omega|}}\exp\left[-2\omega^2\varepsilon(1-\mu)t + i\omega x - h\left(\frac{|\omega|}{4\nu^2\varepsilon}\right)^{1/4}\right]\times\sin\left[t\sqrt{|\omega|\varepsilon} - h\left(\frac{|\omega|}{4\nu^2\varepsilon}\right)^{1/4}\right]; \quad \int_{|\omega|>\omega_0} = \int_{-\infty}^{-\omega_0} + \int_{\omega_0}^{\infty}$$

with an error $O(\varepsilon^{1/4})$ smaller than unity.

If the atmospheric wind is absent at the initial instant, $t' = 0$ rapidly reaches the value $v'(x')$ and then remains constant with respect to time, formula (1.5) can be simplified. Neglecting the variation of function v' from 0 to $v'(x')$ over a small time interval, we obtain

$$\zeta = \frac{i\rho}{\sqrt{2\pi}}\int_{|\omega|>\omega_0(t)} V(\omega)\,K(\omega,t,x)\,d\omega \tag{1.6}$$

Separating in the Fourier transform $V(\omega)$ the oscillating part and including it in the kernel $K'(\omega, t, x)$ we reduce the determination of the free surface shape ζ at time independent wind to the investigation of integrals of the form

$$S(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0(t)}^{\infty} \frac{f(\omega)}{\sqrt{\omega\varepsilon}} E(\omega,t) \times \exp\left[i\left(\omega x + t\sqrt{\omega\varepsilon} - h\left(\frac{\omega}{4\nu^2\varepsilon}\right)^{1/4}\right)\right] d\omega \tag{1.7}$$

$$E(\omega,t) = \exp\left[-2\omega^2\varepsilon(1-\mu)t - h\left(\frac{\omega}{4\nu^2\varepsilon}\right)^{1/4}\right]$$

in which function $f(\omega)$ is of a nonoscillating kind.

Since the index of the oscillating exponent contains the large parameter $\varepsilon^{-1/4}$, it is possible to apply the method of stationary phase for estimating these integrals. We obtain:

1°. When

$$x_0 = \frac{4\nu t^3\varepsilon^2}{h^2} \ll x < \left(\frac{k\varepsilon^2 t^3 h}{128}\sqrt{\frac{k}{\nu}}\right)^{1/4}; \qquad t \ll \frac{h}{2}\left(\frac{1}{4\nu\varepsilon^2}\sqrt{\frac{k}{\nu}}\right)^{1/2} \tag{1.8}$$

$$S = \frac{2}{\sqrt{3x\varepsilon}} f\left(\frac{\xi_0^4}{\varepsilon}\right) E \exp\left[i\left(t\xi_0^2 - \frac{h\xi_0}{\sqrt{2\nu\varepsilon}} + \frac{x}{\varepsilon}\xi_0^4 + \frac{\pi}{4}\right)\right]$$

$$\xi_0 = \left(\frac{\varepsilon h^2}{32\nu x^2}\right)^{1/6}, \quad E = \exp\left[-\frac{1-\mu}{32\nu}\left(\frac{g^2 t'^3}{4\nu_1 y^8}\right)^{1/3} - \left(\frac{g h'^3}{16\nu_1^2 y}\right)^{1/3}\right], \quad y = \frac{x'}{h'}$$

Here the constraints on x and t follow (with allowance for (1.4)) from the inequality $\xi_0 > (\varepsilon\,\omega_0(t))^{1/4}$ which implies that the stationary point belongs to the integration interval.

2°. When $|x| \ll x_0,\ t < t_0$

$$S = \frac{4\xi_0}{\varepsilon\sqrt{2t}} f\left(\frac{\xi_0^4}{\varepsilon}\right) E \exp\left[i\left(t\xi_0^2 - \frac{h\xi_0}{\sqrt{2\nu\varepsilon}} + \frac{x}{\varepsilon}\xi_0^4 + \frac{\pi}{4}\right)\right] \tag{1.9}$$

$$\xi_0 = \frac{h}{2t\sqrt{2\nu\varepsilon}}, \quad E = \exp\left[-\frac{(1-\mu)\nu_2 h'^8}{2^{11} g^2 t'^7 \nu_1^4} - \frac{h'^2}{4\nu_1 t'}\right]$$

3°. The constraints in (1.8) and (1.9) imply that as $|x| \to \infty,\ t \not\to \infty$ the stationary point of the phase function of integral (1.7) is outside the integration interval. Hence, when $|x| \to \infty,\ t \not\to \infty$ it is necessary to derive the asymptotics of integral (1.7) by integrating by parts. This yields

$$S = \frac{i}{x\sqrt{2\pi\omega_0\varepsilon}} f(\omega_0) E \exp\left[i\left(\omega_0 x + t\sqrt{\omega_0\varepsilon} - h\left(\frac{\omega_0}{4\nu^2\varepsilon}\right)^{1/4}\right)\right] \tag{1.10}$$

$$E = \exp\left[-\frac{(1-\mu)h'^2}{2k\nu_1 t'} - \left(\frac{g h'^5}{8\nu_1^2 t'\sqrt{k\nu_1\nu_2}}\right)^{1/2}\right], \quad x \gg 1,\ t \not\to \infty$$

4°. When $|x| \to \infty,\ t \to \infty$ it is possible to disregard the last term in the phase function of integral (1.7). The method of stationary phase then yields

$$1 \ll x < t\varepsilon\left(\frac{t\sqrt{\nu k}}{2h}\right)^{1/2}, \quad t \gg 1 \tag{1.11}$$

$$S(-x,t) = \sqrt{\frac{2}{\varepsilon x}} f(\eta_0) E \exp\left[-i\left(x\eta_0 + \frac{h}{2}\sqrt{\frac{t}{\nu x}} + \frac{\pi}{4}\right)\right]$$

$$\eta_0 = \frac{t^2\varepsilon}{4x^2}, \quad E = \exp\left[-\frac{(1-\mu)\nu_1 g t'^5}{8x'^4} - \frac{h'}{2}\left(\frac{g t'}{\nu_1 |x'|}\right)^{1/2}\right]$$

When in cases 1°, 2°, and 4° x is outside the respective interval, S is asymptotically zero. The method of integration by parts and that of stationary phase yield the same error: formulas (1.8)—(1.11) are valid under the condition that the quantity $O(\varepsilon^{1/4})$ is smaller than unity and can be neglected. Note that case 4° with $|x| \to \infty,\ t \to \infty$ defines the behavior of integral (1.7) but not of the formula for the shape of interface of the fluid media, since formulas (1.6) and (1.7) are constructed on the basis of (1.2), the formula which is not valid when $s = 0$ (i.e. $t = \infty$). It is, however worthwhile to consider formula (1.11), since under certain conditions ($\mu \to 0,\ h \to 0$) it becomes the formula known in the theory of fluid motion under the action of stresses at its surface /6,7/.

5°. It remains to consider the case when $|x| \not\to \infty,\ t \to \infty$, in which formula (1.6) for constant wind currents in the atmosphere is no longer valid, since it results in the contradictory conclusion of cessation of motion of the fluid. To determine the state of fluid in the neighborhood of the considered constant flow $v(x)$ in the atmosphere as $t \to \infty$ we revert to the input boundary value problem which we shall solve in the steady state, setting $\partial/\partial t = 0$.

As the result, we obtain for the form of the interface of fluid media with time independent air flow the following exact expression:

$$\zeta(x) = -\frac{2\mu h}{\pi(1-\rho)}\int_{-\infty}^{\infty} v(\xi)K(x-\xi)\,d\xi, \qquad K(x) = \int_0^{\infty}\frac{\omega^2(\mathrm{sh}\,\omega h + \mu\,\mathrm{ch}\,\omega h)\sin\omega x\,d\omega}{\mathrm{ch}\,2\omega h + \mu\,\mathrm{sh}\,2\omega h - 2\omega^2h^2 - 4\mu\omega h - 1}$$

When the interaction between water and air takes place at 20°C and normal atmospheric pressure, then $\mu = 0{,}018$. Because of this it is possible to disregard in the expression for kernel $K(x)$ terms with the coefficient μ. Further simplifications are linked with the use of parameter h. If $h \ll 1$ (wind directly at the water surface), it is possible to disregard $2\omega^2h^2$ in the denominator of integral $K(x)$. Then

$$\zeta = -\frac{\pi^3\mu}{4h^2(1-\rho)}\int_{-\infty}^{\infty} v(x-\xi)\,\mathrm{sh}\frac{\pi\xi}{2h}\,ch^{-3}\frac{\pi\xi}{2h}\,d\xi, \quad h \ll 1 \tag{1.12}$$

which is correct to $O(\max(\mu, h^2))$.

If $h \gg 1$ (wind high above the water surface), it is possible to disregard $2\omega^2h^2 + 1$ in the denominator of integral $K(x)$ and replace hyperbolic functions by exponential ones. Then

$$\zeta = \frac{\mu h}{\pi(1-\rho)}\int_{-\infty}^{\infty} v(x-\xi)\,\xi(\xi^2 - 3h^2)(\xi^2 + h^2)^{-3}\,d\xi, \quad h \gg 1 \tag{1.13}$$

which is correct to $O(\max(\mu, h^{-1}))$.

The damping increment of E in formulas (1.8)—(1.11) is expressed in terms of dimensional variables (1.1). Analysis of this increment and of the constraints on x and t at which formulas (1.8)—(1.11) are valid leads to such conclusions. The first initial and the second stages, respectively, at $(t \ll t_0)$ and $(t < t_0)$ of wave formation are of short duration. When the wind is at an altitude of several tens of meters the initial stage lasts only seconds and the second, several minutes. For lower altitude wind these times are shorter. Formulas (1.8) and (1.9) are, nevertheless, interesting as providing an insight into the wave formation mechanism. They are qualitatively different. For instance, at the initial stage the time acts as a brake, while in the second it stimulates the development of the wave process. The viscosity of water impedes wave formation, while the viscosity of air sustains wave formation throughout the interaction of wind and ocean. The effect of gravity on the wave amplitude is interesting. During the initial stage $(t \ll t_0)$ it hinders wave generation, while in the second, $t < t_0$, it increases the wave amplitude. During the propagation stage $(x \gg 1)$ (1.10) and damping $(x \to \infty,$ $t \to \infty)$ (1.11) gravitation again decreases the amplitude. A summary of the effects of various characteristics of the wave process on the damping decrement is tabulated below (the signs plus and minus denote, respectively, amlification and weakening, zero means absence of any effect)

Period	$t \ll t_0$	$t < t_0$	$t > t_0$	$t \to \infty$
Time	−	+	+	−
Distance from source	+	0	−	+
Wind altitude	−	−	−	−
Gravitation	−	+	−	−
Viscosity: of water	−	−	+	−
of air	+	+	+	+
Density: of water	−	−	−	−
of air	+	+	+	+

2. Formulas (1.8) and (1.11) are valid for limited values of t'. Observations show that as $t' \to \infty$ no steady flow in the fluid obtains. An unstable one may be possible. An attempt is made below to determine a stable wave mode in a fluid layer $h_1 < z' < \zeta'$ of depth h_1 induced by constant stresses S' acting along tangents to the perturbed fluid surface $z' = \zeta'$ in the Ox' direction. We locate the origin of a orthogonal Cartesian coordinates x', z' on the unperturbed surface of the fluid $z' = 0$ with the Oz' axis pointing in a direction opposite to that of gravity. At $z' = -h_1$ the fluid velocoty is zero, and at $z' = \zeta'$ the dynamic and kinematic conditions are satisfied at the surface. For the determination of velocity $\mathbf{v} = \{v_x, v_z\}$, the difference $p' - p_*$ between the hydrodynamic and atmospheric pressures (as functions of x', z', t'), and of the surface form $z' = \zeta'(x', t')$ we have the system of Navier—Stokes equations and of boundary conditions

$$\frac{\partial\mathbf{v}}{\partial t'} + (\mathbf{v}\nabla)\mathbf{v} + \frac{1}{\rho}\nabla q' = \nu\nabla^2\mathbf{v}, \ \nabla\mathbf{v} = 0, \ q' = p' - p_* + \rho g z' \tag{2.1}$$

$$p_{nn} = -p_*, \ p_{ns} = S', \ \frac{\partial\zeta'}{\partial t'} + v_x\frac{\partial\zeta'}{\partial x'} = v_z, \quad \mathbf{v} = 0 \quad (z' = -h_1) \tag{2.2}$$

Solution of this problem is sought in the form of a wave defined by the actual wave number ω' (or length $\lambda' = 2\pi/\omega'$) and the phase velocity c' which has to be determined. For this we set

$$\mathbf{v}(x', z', t') = \mathbf{v}(\omega'(x' - c't'), z'), \; q' = q'(x, z') \qquad (2.3)$$
$$\zeta' = \zeta'(x), \; x = \omega'(x' - c't'), \; 0 \leqslant x \leqslant 2\pi$$

In dimensionless variables formulas (2.1) — (2.3) are of the form

$$z = \frac{z' + h_1}{h_1}, \quad c' = \frac{\nu c}{h_1} S, \quad \omega' = \frac{\omega}{h_1}, \quad \zeta = \varepsilon^2 h_1 \zeta \qquad (2.4)$$

$$q' = \varepsilon \frac{\rho\nu^2}{h_1^2} q, \quad v_x = \frac{\nu\varepsilon}{h_1}\left(\frac{S}{\varepsilon} z + u\right), \quad v_z = \frac{\nu\varepsilon}{h_1} w, \qquad \varepsilon = \frac{\nu^2}{g h_1^3}, \quad S = \frac{S' h_1^2}{\rho\nu^2}, \quad \sigma = \frac{S'}{\rho g h_1}$$

In these variables the problem (2.1) — (2.3) has for any ω and c the exact solution $u = w = q = \zeta = 0$ which corresponds to a shear flow with a triangular velocity profile and unperturbed fluid surface. We seek such ω and c for which problem (2.1) — (2.3) has a solution different from the stable one. We determine the dynamic conditions (2.2) by expanding them (at $z = 1 + \varepsilon^2\zeta$) in series in powers of $\varepsilon^2\zeta$, and shall seek the solution in terms of series in positive powers of ε. This yields a recurrent sequence of linear boundary value problems for the coefficients of series. For the series zero terms a homogeneous spectral problem is obtained. We confine the investigation to the latter. Its solution is of the form

$$(u, w, q, \zeta) = a\,[(u_1(z), w_1(z), q_1(z), 1)\, e^{ix} + (\bar{u}_1(z), \bar{w}_1(z), \bar{q}_1(z), 1)\, e^{-ix}] \qquad (2.5)$$

Using methods known in the theory of branching /8/ we can represent the amplitude $2a > 0$ in terms of corrections on ω and c by analyzing the conditions of solvability of inhomogeneous problems for the succeeding coefficients of series in ε. Leaving this relation aside, we would only point out that for fairly small ε there exists, as shown in /8,9/, an auto-oscilation mode (2.5) and, also, that the series in ε are convergent. The coefficients in (2.5) are expressed in terms of w_1 from the linearized for solution $u = w = q = 0$ Navier – Stokes equations in variables (2.4)

$$u_1 = \frac{i w_1'}{\omega}, \quad q_1 = \frac{1}{\omega^2}\{w_1''' - [\omega^2 + i\omega S(z - c)]\, w_1' + i\omega S w_1\} \qquad (2.6)$$

Function w_1 satisfies the equation, and together with function q_1 expressed in conformity with (2.6) in terms of w_1 satisfies the boundary conditions

$$\left[\frac{d^2}{dz^2} - \omega^2 - i\omega S(z - c)\right]\left(\frac{d^2}{dz^2} - \omega^2\right) w_1 = 0 \qquad (2.7)$$
$$w_1 = w_1' = 0 \; (z = 0); \; w_1'' + \omega^2 w_1 = 0$$
$$w_1 = -\sigma\, \frac{i\omega(c - 1)}{1 - 2i\sigma\omega}(q_1 - 2w_1') \quad (z = 1)$$

The substitution of $(-\omega)$ for ω indicates passing to complex conjugation, with formulas (2.5) remain unchanged. This means that it is sufficient to find $\omega > 0$. Then all wave numbers in the zero approximation are of the form $\pm\omega$.

We seek a solution of problem (2.7) in the form of series in positive powers of parameter σ. We confine ourselves to the determination of the zero terms of this series neglecting the quantity $O(\sigma)$ as smaller than unity. For this we obtain from (2.7) the following equations and conditions:

$$w_1 = \frac{1}{\omega}\int_0^z W(x)\,\mathrm{sh}(\omega(z - x))\,dx \qquad (2.8)$$

$$W'' - [\omega^2 + i\omega S(z - c)]\, W = 0 \qquad (2.9)$$

$$W(1) = 0, \; \int_0^1 W(x)\,\mathrm{sh}(\omega(1 - x))\,dx = 0 \qquad (2.10)$$

The fundamental system of solutions of the homogeneous equation that corresponds to (2.9) is expressed in terms of Bessel functions with indices $\pm {}^1/_3$

$$\varphi_1(z) = \varphi_1(z, \omega, c) = \xi J_{-1/3}(\beta_1), \; \varphi_2(z) = \varphi_2(z, \omega, c) = \xi J_{1/3}(\beta_1) \qquad (2.11)$$
$$\beta_1 = \beta_1(z) = \frac{2\xi^3}{3\omega S}, \quad \xi = [\omega^2 + i\omega S(z - c)]^{1/2}$$

The branch whose imaginary part is positive for $z \in [0, \infty)$ is fixed at $\sqrt{z}$. If $z \geqslant 0$, the absolute value of $\sqrt{z}$ is taken. A direct determination of the Wronskian of function (2.11) yields

$$\varphi_1\varphi_2' - \varphi_2\varphi_1' = \frac{3\sqrt{3}}{2\pi} i\omega S \tag{2.12}$$

The general solution of the homogeneous equation (2.9) is

$$W = A\varphi_1(z, \omega, c) + B\varphi_2(z, \omega, c) \tag{2.13}$$

For the determination of constants A and B we introduce (2.13) in (2.10). We have

$$A\varphi_1(1, \omega, c) + B\varphi_2(1, \omega, c) = 0 \tag{2.14}$$
$$AJ^{(1)}(1, \omega, c) + BJ^{(2)}(1, \omega, c) = 0$$

$$J^{(j)}(z) = J^{(j)}(z, \omega, c) = \int_0^z \varphi_j(x, \omega, c) \operatorname{sh}(\omega(z-x))\,dx, \quad j = 1, 2 \tag{2.15}$$

The conditiion of solvability of the homogeneous system (2.14) is defined by the complex equation

$$J^{(2)}(1, \omega, c)\,\varphi_1(1, \omega, c) = J^{(1)}(1, \omega, c)\,\varphi_2(1, \omega, c) \tag{2.16}$$

which is used for determining the actual wave number ω and phase velocity c in the zero approximation.

From (2.8), (2.13), (2.15), and (2.16) we obtain the integral representation of coefficients in formula (2.5)

$$u_1 = -\frac{i}{\omega^2}[AJ^{(1)'}(z) + BJ^{(2)'}(z)], \quad \zeta_1 = 1 \tag{2.17}$$
$$w_1 = \frac{A}{\omega} J^{(1)}(z) + \frac{B}{\omega} J^{(2)}(z), \quad q_1 = AQ_1(z) + BQ_2(z)$$
$$Q_j(z) = \frac{1}{\omega^2}\{\varphi_j(z) - iS[(z-c)J^{(j)'}(z) - J^{(j)}(z)]\}, \quad j = 1, 2$$

For the determination of A and B we have formulas (2.14), (2.16),

$$B = -\frac{\varphi_1(1)}{\varphi_2(1)} A \tag{2.18}$$

and the dynamic condition $p_{nn} = -p_*$ on surface $z' = \zeta'$ which in variables (2.4) yields $\zeta = q + 2\omega\, \partial u/\partial x\ (z = 1)$ accurate to $O(\sigma)$. Introducing here the quantities of (2.5) and taking into account (2.17) and that $\zeta_1 = 1$, we obtain the second relationship between A and B, which with the use of (2.8), (2.12), and (2.16) reduces to

$$A = \left\{-\frac{3iS\sqrt{3}}{2\pi\omega\varphi_2(1)} - \left[\frac{2}{\omega} + \frac{iS(1-c)}{\omega^2}\right]\left[J^{(1)'}(1) - \frac{\varphi_1(1)}{\varphi_2(1)} J^{(2)'}(1)\right]\right\}^{-1} \tag{2.19}$$

3. The analysis of solutions of Eq. (2.16) for frequencies is carried out for $S \gg 1$. We set

$$\omega = S\alpha, \ \xi = S\eta, \ \beta_1 = S\beta, \ \beta = \beta(z) = \frac{2\eta^3}{3\alpha} \tag{3.1}$$
$$\eta = \eta(z) = \sqrt{\alpha^2 + i\alpha(z-c)}$$

and apply to (2.11) the asymptotic formulas for Bessel functions at large values of the argument. It is then possible to set

$$\varphi_1(z) = \frac{C}{\sqrt{\eta(z)}} \cos S\beta(z), \ \varphi_2(z) = \frac{C}{\sqrt{\eta(z)}} \sin S\beta(z), \ C = \sqrt{\frac{3\sqrt{3}}{2\pi}\alpha S} \tag{3.2}$$

The constant C is chosen so that the Wronskian of the system of functions (3.2) is the same as $S \to \infty$, as the expression in (2.17). Substituting (3.2) into (2.15) we obtain the integrals

$$J^{(1)}(1) = -\frac{C}{S}\left[\frac{\eta_0 \sin S\beta_0 + i\alpha\cos S\beta_0}{2\alpha c\sqrt{\eta_0}} e^{\alpha S} + \frac{i\cos S\beta_1}{(1-c)\sqrt{\eta_1}}\right] \tag{3.3}$$
$$J^{(2)}(1) = \frac{C}{S}\left[\frac{\eta_0\cos S\beta_0 - i\alpha \sin S\beta_0}{2\alpha c\sqrt{\eta_0}} e^{\alpha S} - \frac{i \sin S\beta_1}{(1-c)\sqrt{\eta_1}}\right]$$

$$\eta_0 = \eta(0),\ \eta_1 = \eta(1),\ \beta_0 = \beta(0),\ \beta_1 = \beta(1),\ S \to \infty$$

determined asymptotically as $S \to \infty$.

Substituting (3.2) and (3.3) into (2.16) we obtain the frequency equation (3.4)

$$\operatorname{ctg} \frac{2S(\eta_0^3 - \eta_1^3)}{3\alpha} = \frac{i\alpha}{\eta_0} \tag{3.4}$$

which implies that

$$\eta_0^3 - \eta_1^3 = \alpha\mu_k - \frac{3\alpha}{2S} \operatorname{arctg} \frac{i\alpha}{\eta_0},\ \mu_k = \frac{3\pi(2k+1)}{4S},\ k = 0, \pm 1, \pm 2, \ldots \tag{3.5}$$

We set

$$X = (\alpha, c),\quad \pi(X) = \eta_0^3 - \eta_1^3 - \alpha\mu_k,\quad R(X) = \frac{3\alpha}{2} \operatorname{arctg} \frac{i\alpha}{\eta_0}$$

Then (3.5) reduces to the equation

$$\pi(X) + S^{-1}R(X) = 0 \tag{3.6}$$

Such equations were obtained in /10/, whose results made possible to establish that, when functions $\pi(X)$ and $R(X)$ have first and second order derivatives with respect to α and c and at point X_0, where $\pi(X_0) = 0$ the first derivatives of π with respect to α and c are not simultaneously zero, then for fairly large S the solution of Eq. (3.6) is close to that of equation $\pi(X) = 0$ and can be derived using the Newton–Kantorovich method with initial approximation X_0. Existence of the necessary derivatives is ensured by that the function $\operatorname{arctg}(i\alpha/\eta_0)$ is analytic with respect to its argument, since $|i\alpha/\eta_0| < 1$ when $c \neq 0$. Function $\pi(X)$ also has limited derivatives with respect to α and c in any finite domain of variations of these variables not containing point $\alpha = c = 0$. Hence in the case of fairly large S we have instead of (3.6) the equation

$$\eta_0^3 - \eta_1^3 - \alpha\mu_k = 0,\ \eta_0 = \sqrt{\alpha^2 - i\alpha c},\ \eta_1 = \sqrt{\eta_0^2 + i\alpha} \tag{3.7}$$

whose solution differs from that of the input equation (3.6) by $O(1/S)$ as $S \to \infty$.

To solve Eq. (3.7) we set

$$\eta_0 = {}^1/_2 e^{i\pi/4} \sqrt{\alpha}\,(x - x^{-1}) \tag{3.8}$$

Then from (3.7)

$$3px^4 - x^3 + p = 0,\quad p = \frac{\sqrt{\alpha}}{4\mu_k} e^{-\pi i/4} \tag{3.9}$$

which is to be supplemented by the relation $\eta_0^2 = \alpha^2 - i\alpha c$, from which by virtue of (3.8) we have

$$\alpha = -{}^1/_4 \operatorname{Im}(x^2 + x^{-2}),\ c = {}^1/_2 - {}^1/_4 \operatorname{Re}(x^2 + x^{-2}) \tag{3.10}$$

and the inequality $\operatorname{Im} \eta_0 > 0$ that fixes the branch of root required for eliminating irrelevant roots. With x as the variable it is of the form

$$\operatorname{Im}\left[(x^{-1} - x)e^{i\pi/4}\right] < 0 \tag{3.11}$$

We restrict the investigation of system (3.9)–(3.11) to the following three cases.

1°. $\mu_k \to 0$. In this case the number p in (3.9) is a large parameter. Equation (3.9) has the following roots:

$$x_n = 3^{-1/4} \exp\left(\frac{2n+1}{4}\pi i\right) + O(p^{-1}),\quad n = 0, 1, 2, 3 \tag{3.12}$$

For the determination of n we use condition (3.11) on the basis of which we conclude, with allowance for (3.12) that $\cos(\pi n/2 - \pi/3) > 0$. Hence $n = 0$ and 1. Introducing (3.12) with these n in (3.10) and rejecting negative values of α, we obtain

$$\alpha = \frac{1}{2\sqrt{3}},\quad c = \frac{1}{2},\quad |\mu_k| \ll 1 \tag{3.13}$$

2°. $\mu_k \to -\infty$. Here p^{-1} is a large parameter and

$$\alpha = \sqrt{-\mu_k/2},\ c = {}^1/_2,\ \mu_k \to -\infty \tag{3.14}$$

3°. $\mu_k \to \infty$. System (3.9) — (3.11) has not real solutions for α.

Let us further simplify formulas (2.18) and (2.19) using relations (3.2) and (3.3), similar expressions for the derivatives $J^{(j)}(z)$ when $z = 1$, and frequency equation (3.5). We obtain

$$A = \frac{S\alpha \sin S\beta_1}{4C\sqrt{\eta_1}}, \quad B = -\frac{S\alpha \cos S\beta_1}{4C\sqrt{\eta_1}} \tag{3.15}$$

4. Leaving aside the problem of velocity and dynamic pressure fields throughout the fluid depth, we shall determine only the horizontal velocity at the fluid surface and the hydrodynamic pressure on the ground. From (2.17), (3.3), (3.2), and (3.15) we have

$$u_1(1) = -\frac{i}{2\alpha S}, \quad q_1(0) = -\frac{i}{4\alpha S}\left(\frac{2\alpha - i}{2\alpha + i}\right)^{\prime\prime} \tag{4.1}$$

Now using formulas (2.3) — (2.5), (3.13), (3.14), and (4.1) we obtain the formula for the fluid surface shape

$$\zeta' = a' \cos x, \quad x = \frac{2\pi}{\lambda'}(x' - c't') \tag{4.2}$$

where a' is the dimensional analog of $2a$ in (2.5).

The horizontal velocity of fluid at the surface

$$v_x(x', \zeta') = \frac{S'}{\mu}(h_1 + \zeta') + \frac{g\lambda' a'}{4\pi\nu}\sin x \tag{4.3}$$

The hydrodynamic pressure on the ground

$$p'(x', -h_1) = p_* + \rho g h_1 + \frac{\rho g \lambda' a'}{8\pi h_1}\psi(x) \tag{4.4}$$

$$\psi(x) = \begin{cases} \sin x, & \mu_k \to -\infty \\ \cos(x - \pi/3), & |\mu_k| \ll 1 \end{cases}$$

The phase velocity, period, and wave length

$$c' = \frac{S' h_1}{2\mu}, \quad T' = \frac{\lambda'}{c'} \tag{4.5}$$

$$\lambda' = \frac{2\pi\rho\nu^2}{S' h_1 \alpha} = \frac{\pi\nu}{c'\alpha} = \begin{cases} (\pi\nu/c')\sqrt{-2/\mu_k}, & \mu_k \to -\infty \quad \text{(a)} \\ 2\pi\nu\sqrt{3}/c', & |\mu_k| \ll 1 \quad \text{(b)} \end{cases}$$

$$\mu_k = \frac{3\pi\rho\nu^2(2k+1)}{4S' h_1^2} = \frac{3\pi\nu(2k+1)}{8c' h_1}, \quad k = 0, \pm 1, \pm 2, \ldots$$

The following observations are based on formulas (4.2) — (4.5).

The flow is decomposed into two wave systems defined in (4.5) by the lengths (a) and (b). Waves of the (b) type virtually do not differ as to length and period. The number of them is limited, with the number k determined by the condition $|2k+1| \ll 4S' h_1^2/(3\pi\rho\nu^2)$. Owing to this these waves are not always present, particularly over shallow waters. The maximum pressure on the ground, in case of waves (b), lags behind the wave crest in phase by $\pi/3$.

Waves of the (a) type differ as regards their period and length. Their number is infinite. The represent a one-parameter set of waves that propagate in the direction of tangent stresses. Waves of type (a), unlike those of type (b), exist for any k. Their length and period decrease as $k \to -\infty$, when they have the form of ordinary ripple on the water surface.

The following properties of waves (a) and (b) should be noted.

The phase velocity of all waves induced by tangential stresses at the surface of water is the same.

The perturbed velocity on the slope ahead of the wave crest is in the direction of tangential stresses at the surface, while behind the wave crest along the rise the direction of velocity is opposite to that of stresses.

An increase of stresses S' has a smoothing effect on the surface of fluid. since it tends to decrease the length of waves and their /oscillation/ period and to increase the phase velocity. An increase of the fluid depth also decreases the wave length and period while increasing the wave phase velocity. On the other hand, increase of viscosity increases the wave length and period but diminishes the phase velocity. The fluid density has the same effect on waves, a fact that is in accord with experiments. It is known that a sharp decrease of the density of undulating water by covering it with liquid oil weakens its surface oscillations. An interesting peculiarity of waves is that their length, period, and phase velocity are independent of gravity acceleration g.

The obtained results are valid if the three conditions $\varepsilon \ll 1, \sigma \ll 1, S \gg 1$ which by virtue of (2.4) are equivalent to the stipulation that the fluid depth is

$$h_1 \gg \max\left(S' / (\rho g),\ \sqrt{\rho\nu^2 / S'},\ \sqrt[3]{\nu^2 / g}\right)$$

are satisfied.

The condition that $S \gg 1$ indicates that auto-oscillations (2.5) are investigated in the supercritical mode. The question of that mode stability reduces to the investigation of the imaginary part of complex roots c of the dispersion equation (2.16) for a given fixed ω or $\alpha = \omega / S$ as $S \to \infty$. From (3.7) and (3.8) we obtain for the functional dependence of c on α and x the formula $c = {}^1/_2 - i\alpha - (x^2 + x^{-2}) / 4$. Introducing this expression into the wave function $\exp(i\omega(x - ct))$, we find that the auto-oscillations are stable when $\operatorname{Re}(-ic) \leqslant 0$. Hence the stability condition is of the form $\operatorname{Im}[-i\alpha - (x^2 + x^{-2}) / 4] \leqslant 0$. From this by virtue of (3.16) we have $\alpha \geqslant \alpha_0$, where α_0 is defined in (3.13) and (3.14), as $(\mu_k \to 0)$ and $(\mu_k \to -\infty)$, respectively. This result in dimensional variables in (2.4) indicates that the auto-oscillations (2.5) are stable, if the wave number ω' satisfies conditions

$$\omega' \geqslant \frac{S' h_1^3}{2\rho\nu^2\sqrt{3}}, \quad \mu_k \to 0$$

$$\omega' \geqslant \frac{S' h_1^3}{\rho\nu^2}\sqrt{-\frac{\mu_k}{2}}, \quad \mu_k \to -\infty$$

This shows that the increase of depth and intensity of stresses at the surface, as well as the decrease of density and viscosity of fluid reduce the wave number range in which auto-oscillations (2.5) are stable as $S \to \infty$.

REFERENCES

1. TITOV L.F., Wind-induced Waves. Leningrad, Gidrometeoizdat, 1969.

2. CHERKESOV L.V., Hydrodynamics of Surface and Internal Waves. Kiev, NAUKOVA DUMKA, 1976.

3. SHKADOV V.Ia., On the formation of waves on the surface of viscous fluid under the action of tangential stress. Izv. Akad. Nauk.SSSR, MZhG, No.3, 1970.

4. DITKIN V.A. and PRUDNIKOV A.P., Handbook of Operational Calculus. Moscow, VYSSHAIA SHKOLA, 1965. (See also in English, Pergamon Press, Book No. 09629, 1962).

5. LEBEDEV N.N., Special Functions and Their Application. Moscow, GOSTEKHIZDAT, 1953.

6. POTETIUNKO E.N. and SRUBSHCHIK L.S., Asymptotic analysis of wave motions of a viscous fluid with a free boundary. PMM, Vol.34, No.5, 1970.

7. ZAVOLZHENSKII M.V. and TERSKOV A.Kh., Axisymmetric waves on the surface of viscous fluid. PMM, Vol.43, No.2, 1979.

8. IUDOVICH V.I., The onset of auto-oscillations in a fluid. PMM Vol.35, No.4, 1971.

9. IUDOVICH V.I., Investigation of auto-oscillations of a continuous medium, occurring at loss of stability of stationary mode. PMM, Vol.36, No.3, 1972.

10. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis. English translation, Pergamon Press, Book No. 10091, 1964.

Translated by J.J.D.